General Topology and its Applications 2 (1972) 277–285. North-Holland Publishing Company

1-FLG COMPLEXES ARE TAME IN 3-MANIFOLDS

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Received 1 June 1971

Abstract: The condition 1-FLG has been used to tame finite graphs. We extend this condition so that it applies to all topological complexes in a 3-manifold. We show that a topological complex which is a closed subset of a 3-manifold is tame if and only if its complement has 1-FLG at each point of K.

AMS Subj. Class.: Primary 5705, 5478

Tame 1-FLG 3-manifold

1. Introduction

Let K be a topological complex in a 3-manifold M^3 and $p \in K$. The *subdivision* K_p of K is defined by: $K = K_p$ if p is a vertex of K, and if p is not a vertex of K then K_p has exactly one more vertex than K, namely p. We say that $M^3 \backslash K$ *has free local fundamental groups at* p (1-FLG at p) if for each sufficiently small open connected set U containing p there is an open set V such that $p \in V \subset U$ and if W is any open connected set such that $p \in W \subset V$ and $W \cap \Delta$ is connected for each simplex Δ of K_p then for each non-empty component W' of $W \backslash K$ the image of the inclusion homomorphism $\iota_*: \pi(W') \to \pi(U')$ is a free group on $m-1$ generators where U' is the component of $U \backslash K$ containing W' and m is the number of components of $\mathrm{st}(p) \backslash \{p\}$ that meet $\mathrm{Cl}(W')$.

The free group on zero generators is understood to be the trivial group. The complement of a topological complex always has 1-FLG at each point in the interior of the topological complex. For, we may pick V to lie in K. Then for any $W \subset V$ there are no non-empty components of $W \backslash K$. We prove the following:

Theorem 1.1. Suppose K is a topological complex which is a closed subset of a 3-manifold M^3. Then K is tame iff $M^3\backslash K$ has 1-FLG at each point of K.

The next proposition shows that if K is a finite graph the above condition is equivalent to "$M^3\backslash K$ has 1-FLG at each point p of K" as defined in [10].

Proposition 1.2. Suppose K is an n-frame, K is the star of the vertex p of K and, K is topologically embedded in M^3. Then $M^3\backslash K$ has 1-FLG at p iff for each sufficiently small open set U containing p there is an open set V such that $p \in V \subset U$ and if W is any connected open set such that $p \in W \subset V$, then the image under the inclusion homomorphism i_*: $\pi_1(W\backslash K) \to \pi_1(U\backslash K)$ is a free group on $n-1$ generators.
Proof. If W_1 is any "small" open connected set containing p, H the component of $W_1 \cap K$ that contains p and $W = W_1\backslash(K\backslash H)$ then $W \cap \Delta$ is connected for each Δ in K_p and $W\backslash K = W_1\backslash K$.

For K a finite graph, Doyle [7] and Debrunner and Fox [5] have shown that a necessary condition for K to be tame is that $M^3\backslash K$ have 1-FLG at each point of K. Cannon has shown in [5] that this condition is also sufficient.

The condition 1-FLG includes 1-ALG. For, if K is a 1-manifold, $M^3\backslash K$ has 1-ALG at a point p of K iff $M^3\backslash K$ has 1-FLG at p (see [10]).

For K a 2-manifold with boundary which is closed in M^3, Bing demonstrated in [3] that $M^3\backslash K$ is 1-LC at each point of K is necessary and sufficient for K to be tame.

Proposition 1.3. Suppose K is a topological complex in a 3-manifold M^3. Suppose p is in the topological boundary of K and the link of p in K_p is connected. Then $M^3\backslash K$ has 1-FLG at p iff $M^3\backslash K$ is 1-LC at p.
Proof. Since link (p) in K_p is connected, $\mathrm{st}(p)\backslash\{p\}$ has only one component.

As a corollary to Theorem 1.1 we may extend [3] as follows:

Corollary 1.4. Suppose K is a topological complex which is a closed subset of a 3-manifold M^3. Suppose K is not an arc and the link of each vertex of K is connected. Then K is tame iff $M^3\backslash K$ is 1-LC at each point of the topological boundary of K in M^3.

2. Definitions

For a vertex v of a topological complex we denote the star of v by $\mathrm{st}(v)$ and the link of v by $\mathrm{lk}(v)$.

Let $x \in L$, where L is a 2-manifold with boundary in a 3-manifold M^3. The local separation theorem [1, §2, Corollary 2] yields: For every $\epsilon > 0$, there is an ϵ-neighborhood N of x in M^3 such that $N \setminus L$ has two components O_1 and O_2. If $x \in \mathrm{Bd}\, L$, then $O_2 = \emptyset$. If $x \in \mathrm{Int}\, L$, then O_1 and O_2 are non-empty.

We say that $U \subset M^3$ is a *1-sided neighborhood* of x for L if there is a neighborhood N of x from the local separation theorem such that $O_1 \cup (N \cap L) \subset U$ and $O_2 \cap U = \emptyset$. Let $x \in L$, U a 1-sided neighborhood of x and N, O_1 and O_2 given for U. We say that $M^3 \setminus L$ is 1-LC *on the U side of L at x* if for every $\epsilon > 0$ there is a neighborhood $N(\epsilon)$ of x from the local separation theorem such that $N(\epsilon) \subset N$ and any simple closed curve in $O_1(\epsilon)$ can be shrunk to a point in $M^3 \setminus L$ on a set of diameter less than ϵ. If $x \in \mathrm{Int}\, L$ then L is said to be *locally tame from the U side at x* if x has a 3-cell neighborhood in U. It follows from [3, Theorems 4, 8] that L is locally tame from the U side at x if L is 1-LC on the U side at each point in a neighborhood of x.

Suppose $N \subset M^3$ is a topological complex which contains no 3-simplexes, N is the star of one of its vertices v, and n = the number of vertices of $\mathrm{lk}(v)$ in N. Then a *reduction* of N is a sequence of subcomplexes of N, $\{C_j\}$, defined by: Let $C_1 = N$. If $\mathrm{lk}(v)$ in C_j contains a simple closed curve S_j, let A_j be a 1-simplex of S_j, let Δ_j = the join of v and A_j in C_j and let $C_{j+1} = C_j \setminus \mathrm{Int}\, \Delta_j \setminus \mathrm{Int}\, A_j$. If $\mathrm{lk}(v)$ in C_j contains no simple closed curves and $\mathrm{lk}(v)$ in C_j contains a 1-simplex A_j, let Δ_j = the join of v and A_j in C_j and $C_{j+1} = C_j \setminus \mathrm{Int}\, \Delta_j \setminus \mathrm{Int}\, A_j$. If $\mathrm{lk}(v)$ in C_j contains no 1-simplexes, let $C_{j+1} = C_j$. There is a first number i such that $\mathrm{lk}(v)$ in C_i contains no simple closed curves. The *reduced* complex C of N is $C = C_i$. The join of v and the vertices of $\mathrm{lk}(v)$ is an n-frame T and $T = C_m$, for some m.

Suppose U and W are open and connected in E^3 and $W \subset U$. Then a finite connected graph $G \subset W$ is a *spine for the pair* (U, W) if the inclusion induced homomorphism of $\pi(G)$ into $\pi(U)$ is an isomorphism onto the image of the inclusion induced homomorphism of $\pi(W)$ into $\pi(U)$.

3. Lemmas

Lemma 3.1. Suppose U and W are open and connected in E^3 and $W \subset U$. Then the image of $i_*: \pi(W) \to \pi(U)$ is a free group on $m - 1$ generators iff there is a spine G for the pair (U, W) whose Euler characteristic, $E(G) = 2 - m$.

Proof. Suppose the image of i_* is a free group on $m - 1$ generators. There is a wedge G of simple closed curves such that the inclusion induced homomorphism $j_*: \pi(G) \to \pi(U)$ maps onto Im i_* and $E(G) = 2 - m$. Since the rank of $\pi(G) = 1 - E(G) = m - 1$ and a homomorphism of a free group of finite rank onto a free group of the same rank has trivial kernel [9, Theorem 2.13, p. 109], G is a spine.

The converse is immediate.

Lemma 3.2. (McMillan). Suppose K is a topological complex in E^3 such that K is the star of a vertex v of K and K contains no 3-simplexes. Let T be the n-frame in K that is the join of v and the vertices of $\mathrm{lk}(v)$ in K. Suppose $K\backslash(\{v\} \cup \mathrm{lk}(v))$ is locally polyhedral. Suppose also that for each open connected set X containing v there exist open connected sets U, V and V_1 such that $v \in V_1 \subset V \subset U \subset X$ and if W is any open connected set such that $v \in W \subset V_1$ and $W \cap \Delta$ is connected for each simplex Δ in K then the images of i_*j_* and j_* are free groups on $n-1$ generators where i_* and j_* are the inclusion induced homomorphisms

$$\pi(W\backslash T) \xrightarrow[j_*]{} \pi(V\backslash T) \xrightarrow[i_*]{} \pi(U\backslash T).$$

Then T is locally tame at v.

Proof. A proof may be adapted from the one given by McMillan for Lemma 3 in [10]. Let Σ^* be the collection defined by McMillan in that proof with the additional condition: $\Sigma \cap \Delta$ is connected for each simplex Δ of K.

4. Proof of Theorem 1.1.

Suppose $M^3\backslash K$ has 1-FLG at each point of K. Let H be the union of the 1- and 2-skeleton of K. Then $M^3\backslash H$ has 1-FLG at each point of H. For, let $p \in H$ and let $\{B_i\}$ be the collection of 3-simplexes in the subdivision K_p of K that contain p. Let U be any sufficiently small connected open set containing p and V given for U. We may choose a smaller V if necessary so that $V \cap (\mathrm{lk}(p) \text{ in } K_p) = \emptyset$. For each B_i, there is a set $O_i \subset U$ which is open in B_i, $p \in O_i$, $O_i \cap \mathrm{Bd}\, B_i \subset V \cap \mathrm{Bd}\, B_i$ and $\pi(O_i \cap \mathrm{Int}\, B_i) = 0$. Let $V_1 = (V\backslash \cup \{B_i\}) \cup \{O_i\}$. Suppose W is any open connected set such that $p \in W \subset V_1$ and $W \cap \Delta$ is connected for each Δ in H. Let W' be any component of $W \backslash H$. If W' lies in some O_i then the image of the inclusion

$$\pi(W') \to \pi(O_i \cap \operatorname{Int} B_i) \to \pi(U_i')$$

is zero. If W' does not lie in any O_i then W' is a component of $W \backslash K$.

Let v be a vertex of H and $N = \operatorname{st}(v)$ in H. The space $N \backslash (\{v\} \cup \operatorname{lk}(v))$ is locally tame. For, let $p \in N \backslash (\{v\} \cup \operatorname{lk}(v))$. If p lies in the interior of some 2-simplex or if p lies in a 1-simplex that is an edge of exactly one 2-simplex then N is locally tame at p by Proposition 1.3 and [3]. If p lies in a 1-simplex that is an edge of no 2-simplex than N is locally tame at p by Proposition 1.2 and [4].

Suppose p lies in a 1-simplex that is an edge of two or more 2-simplexes, Δ_i, $i = 1, ..., r$. Let D be a 3-cell containing p in its interior such that $D \cap (\operatorname{lk}(p) \text{ in } K_p) = \emptyset$. Let U be any open connected set in Int D that contains p and meets each simplex of N in a connected set. Let $\{U_i\}$, $i = 1, ..., r$, be the collection of components of $U \backslash N$ and $\operatorname{Cl}(U_i)$ denote the closure of U_i in U. Thus $\operatorname{Cl}(U_i)$ meets the interior of exactly two 2-simplexes of N and is a 1-sided neighborhood of p for the union of these two 2-simplexes. The components of U then define the "sides" of N at p. It follows from Proposition 1.2 that $M^3 \backslash N$ is 1-LC on each "side" of N at p. Thus N is locally tame from each of its "sides" at p [3]. That is, p has a 3-cell neighborhood B_i in $\operatorname{Cl}(U_i)$, for each $i = 1, ..., r$.

Let O be an open neighborhood of p in M^3 contained in the union of the B_i's. Let k be a homeomorphism of $I = [0,1]$ into the intersection of O and the 1-simplex such that $k(\frac{1}{2}) = p$. For each $i = 1, ..., r$, there is a homeomorphism k_i of I^2 into $O \cap \Delta_i$ that extends k. For each pair of numbers ij such that Im k_i and Im k_j lie in the boundary of the same cell B_m (for some m) there is a homeomorphism k_{ij} of I^3 into B_m such that $N \cap k_{ij}(I^3) = k_i(I^2) \cup k_j(I^2)$. The union of the images of the maps k_{ij} is a 3-cell neighborhood Q of p in M^3.

Let P denote a wedge of planes through the origin in $\mathbb{E}^3$ and B the unit ball in E^3. Using the standard technique of extending a homeomorphism on the boundary of a cell to the entire cell, a homeomorphism of $Q \cap N$ to $B \cap P$ may be extended to a homeomorphism of Bd Q onto Bd B and then to a homeomorphism of Q onto B. Thus N is locally tame at p.

Let T be the n-frame which is the join of v and the vertices of $\operatorname{lk}(v)$ in N. We use Lemma 3.2 to show that T is locally tame at v. We may assume that $N \backslash (\{v\} \cup \operatorname{lk}(v))$ is locally polyhedral. Let X be any open connected set containing v. Let B be a 3-cell such that $v \in \operatorname{Int} B \subset B \subset X$ and

$B \cap \mathrm{lk}(v) = \emptyset$. Since $M^3 \backslash H$ has 1-FLG at p, we can show that if U is any sufficiently small open connected set such that $v \in U \subset \mathrm{Int}\, B$ and $U \cap \Delta$ is connected and simply connected for each simplex Δ in N, if V_0 is given for U by the definition of 1-FLG, and if W is any open connected set such that $v \in W \subset V_0$ and $W \cap \Delta$ is connected for each Δ in N, then the image of the inclusion homomorphism of $\pi(W \backslash T)$ into $\pi(U \backslash T)$ is a free group on $n-1$ generators. The procedure is to remove by a reduction of N the interiors of simplexes in N until we reach the n-frame T.

Let C_j be a reduction of N. Suppose $\mathrm{lk}(v)$ in C_1 contains a simple closed curve S_1. Let Δ_1 be the 2-simplex in C_1 given by the reduction. Then there are exactly two components W_1 and W_2 of $W \backslash C_1$ whose closures contain $W \cap \mathrm{Int}\, \Delta_1$. Let U_1 and U_2 be the components of $U \backslash C_1$ that contain W_1 and W_2, respectively. Let m_1 and m_2 denote the number of components of $C_1 \backslash \{v\}$ that meet the closures of W_1 and W_2, respectively. By Lemma 3.1, there exist spines G_1 and G_2 for the pairs (U_1, W_1) and (U_2, W_2), respectively, and $E(G_1) = 2-m_1$ and $E(G_2) = 2-m_2$.

Let $W' = W_1 \cup W_2 \cup (W \cap \mathrm{Int}\, \Delta_1)$ and U' be the component of $U \backslash C_2$ that contains W'. Let $Y = W \cap \mathrm{Int}\, \Delta_1$. There is an arc $A \subset W'$ from a vertex of G_1 to a vertex of G_2 that meets Y at exactly one point and pierces Δ_1 there. Let $G' = G_1 \cup G_2 \cup A$. Then G' is a spine for the pair (U', W'). For, let l be any loop in W'. Then l is homotopic in W' to a loop l_1 in W' that meets Y at a finite number of points and pierces Δ_1 at each point of their intersection. The loop l_1 may be adjusted over to the arc A at the points where l_1 meets the connected set Y. Let B be a subarc of l_1 such that $B \cap Y = \{b\}$ and B pierces Δ_1 at b. It follows from [4] that Y has a bicollar $Y \times (-1, 1)$ such that the diameter of the fibers go to zero as they approach the boundary of Y. There exists $t \in (-1, 1)$, $b_1 \in B \cap (Y \times t)$, $b_2 \in B \cap (Y \times (-t))$ and a homotopy H in $Y \times [-t, t]$ such that $b \in \mathrm{Int}\, [b_1, b_2]$ and H carries the subarc $[b_1, b_2]$ of B into an arc in $(Y \times t) \cup (Y \times (-t)) \cup A$ and H keeps the endpoints of $[b_1, b_2]$ fixed.

Let l_2 be the loop homotopic to l_1 obtained from l_1 by applying a homotopy such as H at each point where l_1 meets Y. It follows from the fact that G_1 and G_2 are spines that l_2 (and hence l) is homotopic in U' to a loop in G'. Thus the inclusion homomorphism of $\pi(G')$ into $\pi(U')$ maps $\pi(G')$ onto the image of $i_*\colon \pi(W') \to \pi(U')$.

The component U' of $U \backslash C_2$ is $U_1 \cup U_2 \cup (U \cap \mathrm{Int}\ \Delta_1)$. By Van Kampen's theorem $\pi(U')$ is the free product of $\pi(U_1)$ and $\pi(U_2)$, also $\pi(G')$ is the free product of $\pi(G_1)$ and $\pi(G_2)$. Since the induced homomorphisms of $\pi(G_1)$ into $\pi(U_1)$ and of $\pi(G_2)$ into $\pi(U_2)$ are one-to-one, the inclusion induced homomorphism of $\pi(G')$ into $\pi(U')$ is also one-to-one.

Thus G' is a spine for (U', W'). The Euler characteristic, $E(G') = E(G_1) + E(G_2) - 1$ and the number of components of $C_2 \setminus \{v\}$ that meet $\mathrm{Cl}(W')$ is

$$m' = m_1 + m_2 - 1 = 2 - E(G_1) + 2 - E(G_2) - 1 = 2 - E(G').$$

We continue this procedure to the reduced complex C.

Let β be the number of 2-simplexes in the reduced complex C, h the number of components of $\mathrm{lk}(v)$ in C and n the number of vertices of $\mathrm{lk}(v)$ in C. Since each component of $\mathrm{lk}(v)$ in C has one more vertex than 1-simplex, $n = \beta + h$. Let $W_h = W \setminus C$. Since $\mathrm{lk}(v)$ in C contains no simple closed curves, W_h is connected. Since $C = C_i$, for some i, we obtain from the above argument that the pair $(U \setminus C, W_h)$ has a spine G_h and $E(G_h) = 2 - h$.

Suppose $C = C_i$ contains a 2-simplex Δ_i. Let A_i be a polyhedral simple closed curve such that $A_i \cap C_i$ is a point of $\mathrm{Int}\,\Delta_i$, A_i pierces Δ_i at this point and $A_i \cap G_h$ is a vertex of G_h. Let $G_{h+1} = G_h \cup A_i$ and $W_{h+1} = W_h \cup (W \cap \mathrm{Int}\,\Delta_i)$. Again it follows that each loop in W_{h+1} is homotopic in $U \setminus C_{i+1}$ to a loop in G_{h+1}. The loop A_i links $\mathrm{Bd}\,\Delta_i$ and thus is not trivial in $U \setminus C_{i+1}$ and not homotopic in $U \setminus C_{i+1}$ to a loop in G_h. A polyhedral singular disk in general position with $\mathrm{Int}\,\Delta_i$ and whose boundary lies in G_h can be cut off of $\mathrm{Int}\,\Delta_i$. So no essential loop in G_h is trivial in $U \setminus C_{i+1}$. It follows that G_{h+1} is a spine for $(U \setminus C_{i+1}, W_{h+1})$. The Euler characteristic, $E(G_{h+1}) = E(G_h) - 1$ and the number of components of $C_{i+1} \setminus \{v\}$ that meet $\mathrm{Cl}(W_{h+1})$ is

$$h + 1 = 2 - E(G_h) + 1 = 2 - E(G_{h+1}).$$

Continue the reduction to the n-frame T. Since $n = \beta + h$, G_n is a spine for the pair $(U \setminus T, W \setminus T)$ and

$$E(G_n) = E(G_h) - \beta = 2 - h - \beta = 2 - n.$$

Thus the number of generators of $\pi(G_n) = 1 - E(G_n) = n - 1$. Thus the image of $i_*: \pi(W \setminus T) \to \pi(U \setminus T)$ is a free group on $n - 1$ generators.

Let V be an open connected set such that $v \in V \subset V_0$ and $V \cap \Delta$ is connected and simply connected for each Δ in N. Let V_1 be given for v by the definition of 1-FLG. Repeating the above argument, we have that U, V and V_1 satisfy the hypothesis of Lemma 3.2. Hence T is locally tame at v.

The 1-skeleton of K is locally tame. Thus the boundary of each 2-simplex is tame [2]. Since the interior of each 2-simplex is locally tame, each 2-simplex is tame [11]. The star of each vertex is tame [8]. Hence, K is locally tame and thus tame [2].

Suppose K is tame and $p \in K$. The proof follows the one given for the lemma in [6, §2]. There is a homeomorphism g of M^3 onto itself that carries K onto a polyhedron. There is a subdivision of M^3 such that $g(K)$ is a subcomplex, $g(p)$ is a vertex and $\mathrm{st}(g(p))$ in $g(K)$ lies in $g(\mathrm{st}(p)$ in $K_p)$.

Let R be the star of $g(p)$ in $g(K)$. We may assume that $g(p)$ is the origin in E^3, $\mathrm{lk}(g(p))$ lies in the boundary of the unit ball B and R is the cone of $\mathrm{lk}(g(p))$ from $g(p)$. For each $0 < t < 1$, let B_t denote the ball of radius t centered at the origin. Let U be any open connected set such that $g(p) \in U \subset \mathrm{Int}\, B$. Let $0 < r < 1$ such that $B_r \subset U$ and let $V = \mathrm{Int}\, B_r$. Suppose W is any open connected set such that $g(p) \in W \subset V$ and $W \cap g(\Delta)$ is connected for each simplex Δ of K_p. Let W' be any non-empty component of $W \backslash R$. Then $W' = W_1 \cap W$, where W_1 is a component of $\mathrm{Int}\, B \backslash R$.

Let $0 < s < 1$ such that $B_s \subset W$. Let $W_r = V \cap W_1$, $W_s = (\mathrm{Int}\, B_s) \cap W_1$ and U' the component of $U \backslash R$ that contains W'. Then $W_s \subset W' \subset W_r \subset U' \subset W_1$. We have that $\pi(W_s) = \pi(W_r) = \pi(W_1) =$ a free group on $m - 1$ generators where $m =$ the number of components of $R \backslash \{g(p)\}$ that meet $\mathrm{Cl}(W_1)$ = the number of components of $R \backslash \{g(p)\}$ that meet $\mathrm{Cl}(W')$. Consider the diagram

$$\pi(W_s) \underset{i_4}{\rightarrow} \pi(W') \underset{i_3}{\rightarrow} \pi(W_r) \underset{i_2}{\rightarrow} \pi(U') \underset{i_1}{\rightarrow} \pi(W_1).$$

The maps $i_4 i_3$ and $i_2 i_1$ are onto isomorphisms. Thus i_3 is onto and i_2 is one-to-one. Thus $\mathrm{Im}\, i_3 i_2 = \mathrm{Im}\, i_2 = \pi(W_r)$. It follows that $M^3 \backslash K$ has 1-FLG at p.

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